On spectral stability of solitary waves of nonlinear Dirac equation on a line

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Abstract

We consider the nonlinear Dirac equation in one dimension (the massive Gross-Neveu model). We explicitly construct solitary wave solutions, and then study the linearization of the equation at a solitary wave. We present numerical simulations and justify them with explicit construction of some of the eigenfunctions. Then we present a WKB-based argument which justifies (but does not prove) the spectral stability of solitary waves of sufficiently small amplitude. We also compare our results with previously known numerical simulations.

1 Introduction

Field equations with nonlinearities of local type are natural candidates for developing tools which are then used for the analysis of systems of interacting equations. Equations with local nonlinearities have been appearing in the Quantum Field Theory perhaps starting with the articles by Schiff [Sch51a, Sch51b], who studied the nonlinear Klein-Gordon equation in his research on the classical nonlinear meson theory of nuclear forces. These papers stimulated development of mathematical tools for tackling such nonlinear equations. Well-posedness for the nonlinear Klein-Gordon equation was addressed by Jörgens [Jör61] and by Segal [Seg63]. Nonlinear scattering was studied by Morawetz and Strauss in [Seg66, Str68, MS72]. Existence of solitary waves in nonlinear Schrödinger equation and Klein-Gordon equation was proved for a large class of nonlinearities in [Str77] and [BL83]. Stability of solitary waves takes its origin from [VK73], where the (spectral) stability of the linearization at a solitary wave for a nonlinear Schrödinger equation has been studied. Orbital stability and instability of solitary waves in nonlinear Schrödinger, Klein-Gordon, and Korteweg – de Vries equations has been extensively studied in [Sha83, SS85, Sha85, Wei86, GSS87, BSS87]. The asymptotic stability of the standing waves in nonlinear Schrödinger equation was proved in certain cases in [Wei85, BP93, SW92, SW99, Cuc01, BS03, CM08].

Much less is known for systems with Hamiltonians that are not sign-definite, such as the nonlinear Dirac equation, such as the Soler model [Sol70], and Dirac-Maxwell system. The existence of standing waves in the nonlinear Dirac equation was studied in [Sol70], [CV86], [Mer88], and [ES95]. The existence of standing waves in the Dirac-Maxwell system is proved in [EGS96] (for $\omega \in (-m, 0)$) and [Abe98] (for $\omega \in (-m, m)$). For an overview of these results, see [ES02]. The local well-posedness of the Dirac-Maxwell system was considered in [Bou96]. The local and global well-posedness of the Dirac equation was further addressed in [EV97] (semilinear Dirac equation in $n = 3$), [Bou00] (interacting Dirac and Klein-Gordon equations in $n = 1$), and in [MNNO05] (nonlinear Dirac equation in $n = 3$). Numerical confirmation of spectral stability of solitary waves of small amplitude is contained in [Chu07]. The asymptotic stability of small amplitude solitary waves in the external potential has been studied in [Bou08].

In this paper, we give numerical and analytical justifications (but not a rigorous proof) of spectral stability of small amplitude solitary wave solutions to the nonlinear Dirac equation in one dimension.

2 Nonlinear Dirac equation

We consider the nonlinear Dirac equation of the form

$$i\partial_t \psi = -i\sum_{j=1}^{n} \boldsymbol{\alpha}_j \partial_{x_j} \psi + \boldsymbol{\beta} g(\bar{\psi}\psi)\psi, \qquad x \in \mathbb{R}^n, \tag{2.1}$$

where

$$\psi = \left[\begin{array}{c} \psi_1 \\ \ldots \\ \psi_N \end{array} \right] \in \mathbb{C}^N \qquad \text{and} \qquad \bar{\psi} = \psi^+ \boldsymbol{\beta},$$

with $\psi^+ = [\bar{\psi}_1, \ldots, \bar{\psi}_N]$ being the Hermitian conjugate of ψ. The Hermitian matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are chosen so that $(-i\sum_{j=1}^{n} \boldsymbol{\alpha}_j \partial_{x_j} + \boldsymbol{\beta}m)^2 = (-\Delta + m^2)I$, where $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ is the Laplace operator and I is the unit matrix. That is, $\boldsymbol{\alpha}_j$ and $\boldsymbol{\beta}$ are to satisfy

$$\boldsymbol{\alpha}_j^2 = I, \qquad \boldsymbol{\beta}^2 = I, \qquad \{\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_k\} = 2\delta_{jk}, \qquad \{\boldsymbol{\alpha}_j, \boldsymbol{\beta}\} = 0.$$

We assume that the nonlinearity g is smooth and real-valued. We denote $m \equiv g(0)$.

When $n = 3$, one can take Dirac spinors ($N = 4$ components). This equation with $n = 3$ and $g(s) = 1 - s$ is the Soler model [Sol70], which has been receiving a lot of attention in theoretical physics in relation to classical models of elementary particles. The case $n = 1$ (when one can take spinors with $N = 2$ components) is known as the massive Gross-Neveu model [GN74, LG75].

In terms of the Dirac $\boldsymbol{\gamma}$-matrices, equation (2.1) takes the form

$$i\boldsymbol{\gamma}^\mu \partial_\mu \psi = g(\bar{\psi}\psi)\psi, \quad \text{where } \boldsymbol{\gamma}^0 = \boldsymbol{\beta}, \quad \boldsymbol{\gamma}^j = \boldsymbol{\beta}\boldsymbol{\alpha}_j, \quad \partial_0 = \partial_t, \quad \partial_j = \partial_{x_j}. \tag{2.2}$$

The nonlinear Dirac equation can be written in the Hamiltonian form as $\dot{\psi} = -i\delta_{\bar{\psi}} E$, where

$$E(\psi, \bar{\psi}) = \int_{\mathbb{R}^3} \left[-i\bar{\psi}\boldsymbol{\beta}\boldsymbol{\alpha}_j \partial_{x_j}\psi + G(\bar{\psi}\psi) \right] dx = \int_{\mathbb{R}^3} \left[\frac{1}{2} \left(-i\bar{\psi}\boldsymbol{\beta}\boldsymbol{\alpha}_j \partial_{x_j}\psi + i\partial_{x_j}\bar{\psi}\boldsymbol{\beta}\boldsymbol{\alpha}_j\psi \right) + G(\bar{\psi}\psi) \right] dx, \tag{2.3}$$

with G being the antiderivative of g such that $G(0) = 0$. Note that $G(\mathscr{X}) = -F(\mathscr{X})$ from [SV86].

We consider the Dirac equation in $\mathbb{R}^1$:

$$i(\partial_t + \boldsymbol{\alpha}\partial_x)\psi = g(\bar{\psi}\psi)\boldsymbol{\beta}\psi. \tag{2.4}$$

As $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, one can take any two of the Pauli matrices. We choose

$$\boldsymbol{\alpha} = -\boldsymbol{\sigma}_2, \qquad \boldsymbol{\beta} = \boldsymbol{\sigma}_3,$$

where the Pauli matrices are given by

$$\boldsymbol{\sigma}_1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \boldsymbol{\sigma}_2 = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \boldsymbol{\sigma}_3 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \tag{2.5}$$

The nonlinear Dirac equation (2.4) takes the form

$$i(\partial_t + (-\boldsymbol{\sigma}_2)\partial_x)\psi = g(\psi^+ \boldsymbol{\sigma}_3 \psi)\boldsymbol{\sigma}_3 \psi. \tag{2.6}$$

We can rewrite this as a system

$$\begin{cases} i\partial_t \psi_1 = \partial_x \psi_2 + g(|\psi_1|^2 - |\psi_2|^2)\psi_1, \\ i\partial_t \psi_2 = -\partial_x \psi_1 - g(|\psi_1|^2 - |\psi_2|^2)\psi_2. \end{cases} \tag{2.7}$$

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3 Solitary wave solutions

We start by demonstrating the existence of solitary wave solution and exploring their properties.

Definition 3.1. The solitary waves are solutions to (2.1) of the form

$$\mathbf{S} = \{\psi(x,t) = \phi_\omega(x)e^{-i\omega t}; \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N), \omega \in \mathbb{R}\}.$$

The following result follows from [CV86].

Lemma 3.2. *Assume that*

$$m \equiv g(0) > 0. \tag{3.1}$$

Let G be the antiderivative of g such that $G(0) = 0$. Assume that for given $\omega \in \mathbb{R}$, $0 < \omega < m$, there exists $\mathscr{X}_\omega > 0$ such that

$$\omega \mathscr{X}_\omega = G(\mathscr{X}_\omega), \qquad \omega \neq g(\mathscr{X}_\omega). \qquad and \qquad \omega \mathscr{X} < G(\mathscr{X}) \quad \text{for } \mathscr{X} \in (0, \mathscr{X}_\omega). \tag{3.2}$$

Then there is a solitary wave solution $\psi(x,t) = \phi_\omega(x)e^{-i\omega t}$, where

$$\phi_\omega(x) = \begin{bmatrix} v(x) \\ u(x) \end{bmatrix}, \qquad v, \, u \in H^1(\mathbb{R}), \tag{3.3}$$

with v, u real-valued.

More precisely, let us define $\mathscr{X}(x)$ and $\mathscr{Y}(x)$ by

$$\mathscr{X} = v^2 - u^2, \qquad \mathscr{Y} = vu. \tag{3.4}$$

Then $\mathscr{X}(x)$ is the solution to

$$\mathscr{X}'' = -\partial_{\mathscr{X}}(-2G(\mathscr{X})^2 + 2\omega^2 \mathscr{X}^2), \qquad \mathscr{X}(0) = \mathscr{X}_\omega, \qquad \mathscr{X}'(0) = 0, \tag{3.5}$$

and $\mathscr{Y}(x) = -\frac{1}{4\omega}\mathscr{X}'(x)$.

Proof. From (2.7), we obtain:

$$\begin{cases} \omega v = \partial_x u + g(|v|^2 - |u|^2)v, \\ \omega u = -\partial_x v - g(|v|^2 - |u|^2)u. \end{cases} \tag{3.6}$$

Taking into account that both v and u are real-valued, we can rewrite (3.6) as the following Hamiltonian system, with x playing the role of time:

$$\begin{cases} -\partial_x u = -\omega v + g(v^2 - u^2)v = \partial_v h(\phi), \\ \partial_x v = -\omega u - g(v^2 - u^2)u = \partial_u h(\phi), \end{cases} \tag{3.7}$$

where

$$h(\phi) = -\frac{\omega}{2}(v^2 + u^2) + \frac{1}{2}G(v^2 - u^2). \tag{3.8}$$

The solitary wave corresponds to a trajectory of this Hamiltonian system such that $\lim_{x \to \pm\infty} v(x) = \lim_{x \to \pm\infty} u(x) = 0$. Since $G(\mathscr{X})$ satisfies $G(0) = 0$,

$$h(v(x), u(x)) \equiv 0. \tag{3.9}$$

Thus,

$$\omega(v^2 + u^2) = G(v^2 - u^2). \tag{3.10}$$

Studying the level curves which solve this equation is most convenient in the coordinates

$$\mathscr{X} = v^2 - u^2, \qquad \mathscr{Z} = v^2 + u^2;$$

see Figure 3. We conclude from (3.10) and Figure 3 that solitary waves may correspond to $|\omega| < m$, $\omega \neq 0$. If $\omega > 0$, then the solitary waves correspond to nonzero v, with u changing its sign.

Remark 3.3. If $\omega < 0$, then the solitary waves correspond to $u \neq 0$, and to v changing the sign.
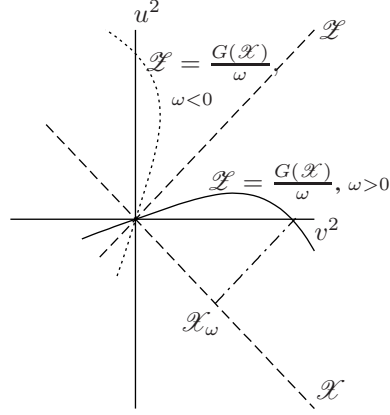
Figure 1: Existence of solitary waves in the coordinates $\mathscr{X} = v^2 - u^2$, $\mathscr{Z} = v^2 + u^2$. Solitons with $\omega > 0$ and $\omega < 0$ correspond to the bump on the v^2 axis and to the dotted bump on the u^2 axis (respectively) in the first quadrant.

The functions $\mathscr{X}(x)$ and $\mathscr{Y}(x)$ introduced in (3.4) should solve

$$\begin{cases} \mathscr{X}' = -4\omega\mathscr{Y}, \\ \mathscr{Y}' = -(v^2 + u^2)g(\mathscr{X}) + \omega\mathscr{X} = -\frac{1}{\omega}G(\mathscr{X})g(\mathscr{X}) + \omega\mathscr{X}, \end{cases} \tag{3.11}$$

with the limit behavior $\lim_{|x|\to\infty} \mathscr{X}(x) = 0$, $\lim_{|x|\to\infty} \mathscr{Y}(x) = 0$. In the second equation in (3.11), we used the relation (3.8) and the identity $h(\phi) = 0$. The system (3.11) can be written as the equation on $\mathscr{X}$:

$$\mathscr{X}'' = -\partial_{\mathscr{X}}(-2G(\mathscr{X})^2 + 2\omega^2\mathscr{X}^2). \tag{3.12}$$

This equation describes a particle in the potential $U_\omega(\mathscr{X}) = -2G(\mathscr{X})^2 + 2\omega^2\mathscr{X}^2$; see Figure 2. Due to the energy conservation (with x playing the role of time), we get:

$$\frac{\mathscr{X}'^2}{2} - 2G(\mathscr{X})^2 + 2\omega^2\mathscr{X}^2 = \frac{\mathscr{X}'^2}{2} + U_\omega(\mathscr{X}) = 0. \tag{3.13}$$

Using the expression for $\mathscr{X}'$ from (3.11), relation (3.13) could be rewritten as

$$0 = \frac{\mathscr{X}'^2}{2} + U_\omega(\mathscr{X}) = 8\omega^2\mathscr{Y}^2 - 2G(\mathscr{X})^2 + 2\omega^2\mathscr{X}^2 = 2\omega^2(4v^2u^2 + (v^2 - u^2)^2) - 2G^2, \tag{3.14}$$

which is equivalent to (3.10).

For a particular value of ω, there will be a positive solution $\mathscr{X}(x)$ such that $\lim_{x\to\pm\infty} \mathscr{X}(x) = 0$ if there exists $\mathscr{X}_\omega > 0$ so that (3.2) is satisfied (see Figure 2). For convenience, we assume that $\mathscr{X}(0) = \mathscr{X}_\omega$, so that $\mathscr{X}(x)$ is an even function.
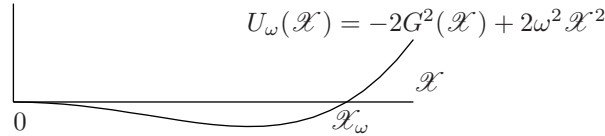

Figure 2: Effective potential $U_\omega(\mathscr{X})$. A solitary wave corresponds to a trajectory which satisfies $\mathscr{X}(0) = \mathscr{X}_\omega$, $\lim_{|x|\to\infty} \mathscr{X}(x) = 0$.

After we find $\mathscr{X}(x)$, the function $\mathscr{Y}(x)$ is obtained from the first equation in (3.11). Knowing $\mathscr{X}(x)$ and $\mathscr{Y}(x)$, one can express $v(x)$ and $u(x)$. $\qquad\square$

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4 Explicit solitary waves in a particular case

As shown in [LG75] for the massive Gross-Neveu model, in the special case of the potential

$$G(\mathcal{X}) = \mathcal{X} - \frac{\mathcal{X}^2}{2}, \tag{4.1}$$

the solitary waves can be found explicitly. Substituting $G(\mathcal{X})$ from (4.1) into (3.13), we get the following relation:

$$dx = -\frac{d\mathcal{X}}{2\sqrt{(\mathcal{X} - \mathcal{X}^2/2)^2 - \omega^2 \mathcal{X}^2}} = -\frac{d\mathcal{X}}{2\mathcal{X}\sqrt{(1 - \mathcal{X}/2)^2 - \omega^2}}. \tag{4.2}$$

We use the substitution

$$1 - \frac{\mathcal{X}}{2} = \frac{\omega}{\cos 2\Theta}, \qquad \mathcal{X} = 2\left(1 - \frac{\omega}{\cos 2\Theta}\right). \tag{4.3}$$

Then

$$dx = -\frac{d\mathcal{X}}{2\mathcal{X}\sqrt{(1 - \mathcal{X}/2)^2 - \omega^2}} = \frac{2\frac{2\omega \sin 2\Theta}{\cos^2 2\Theta} d\Theta}{4(1 - \frac{\omega}{\cos 2\Theta})\sqrt{\frac{\omega^2}{\cos^2 2\Theta} - \omega^2}} = \frac{d\Theta}{\cos 2\Theta - \omega}, \tag{4.4}$$

$$x = \frac{1}{2\kappa} \ln\left|\frac{\sqrt{\mu} + \tan\Theta}{\sqrt{\mu} - \tan\Theta}\right|, \tag{4.5}$$

where

$$\kappa = \sqrt{1 - \omega^2}, \qquad \mu = \frac{1 - \omega}{1 + \omega}. \tag{4.6}$$

$$(\sqrt{\mu} + \tan\Theta)e^{2\kappa x} = \sqrt{\mu} - \tan\Theta, \qquad \tan\Theta = -\sqrt{\mu}\tanh\kappa x. \tag{4.7}$$

Also note that

$$\mathcal{X} = 2\left(1 - \frac{\omega}{\cos 2\Theta}\right) = 2\left(1 - \frac{\omega}{2\cos^2\Theta - 1}\right) = 2\left(1 - \omega\frac{1 + \tan^2\Theta}{1 - \tan^2\Theta}\right), \tag{4.8}$$

and then

$$\mathcal{Y} = -\frac{1}{4\omega}\mathcal{X}' = -\frac{1}{4}\frac{2}{\cos^2 2\Theta}(-2\sin 2\Theta)\frac{d\Theta}{dx} = -\frac{1}{4}\frac{2}{\cos^2 2\Theta}(-2\sin 2\Theta)(\cos 2\Theta - \omega) \tag{4.9}$$

$$= \frac{\mathcal{X}}{2}\tan 2\Theta = \frac{\mathcal{X}}{2}\frac{2\tan\Theta}{1 - \tan^2\Theta} = -\mathcal{X}\frac{\sqrt{\mu}\tanh\kappa x}{1 - \mu\tanh^2\kappa x}. \tag{4.10}$$

Denote

$$\mathcal{L}(x) = v^2(x) + u^2(x). \tag{4.11}$$

Then

$$\mathcal{L}(x) = \frac{2}{\cos 2\Theta}\left(1 - \frac{\omega}{\cos 2\Theta}\right) = 2\frac{1 + \tan^2\Theta}{1 - \tan^2\Theta}\left(1 - \omega\frac{1 + \tan^2\Theta}{1 - \tan^2\Theta}\right).$$

There are the following relations:

$$v = \sqrt{\mathcal{L}}\cos\Theta, \qquad u = -\sqrt{\mathcal{L}}\sin\Theta, \tag{4.12}$$

$$\mathcal{X} = \mathcal{L}\cos 2\Theta, \qquad \mathcal{Y} = -\frac{1}{2}\mathcal{L}\sin 2\Theta. \tag{4.13}$$

Remark 4.1. By (4.7), $\tan\Theta$ changes from $\sqrt{\mu}$ to $-\sqrt{\mu}$ as x changes from $-\infty$ to $+\infty$. Thus, in the limit $\omega \to 1$, when $\mu \to 0$, one has $\mathcal{X} \approx \mathcal{L}$, while $|\mathcal{Y}| \lesssim \mathcal{L}\sqrt{\mu}$.

5 Linearization at a solitary wave

To analyze the stability of solitary waves we consider the solution in the form of the ansatz

$$u(x,t) = (\phi_\omega(x) + \mathcal{R}(x,t))e^{-i\omega t}, \qquad \phi_\omega = \begin{bmatrix} v \\ u \end{bmatrix}. \tag{5.1}$$

Then, by (2.1),

$$i\dot{\mathcal{R}} + \omega\mathcal{R} = -i\sum_j \boldsymbol{\alpha}\partial_{x_j}\mathcal{R} + \boldsymbol{\beta}\left[g((\bar{\phi}_\omega + \bar{\mathcal{R}})(\phi_\omega + \mathcal{R}))(\phi_\omega + \mathcal{R}) - g(\bar{\phi}_\omega\phi_\omega)\phi_\omega\right].$$

The linearized equation on $\mathcal{R}$ is:

$$i\dot{\mathcal{R}} = -i\sum_j \boldsymbol{\alpha}\partial_{x_j}\mathcal{R} - \omega\mathcal{R} + \boldsymbol{\beta}\left[g(\bar{\phi}_\omega\phi_\omega)\mathcal{R} + (\bar{\phi}_\omega\mathcal{R} + \bar{\mathcal{R}}\phi_\omega)g'(\bar{\phi}_\omega\phi_\omega)\phi_\omega\right].$$

Let us write this explicitly. Namely, let $\mathcal{R} = \begin{bmatrix} \rho_1 + i\varsigma_1 \\ \rho_2 + i\varsigma_2 \end{bmatrix}$, with ρ_j, ς_j real-valued. The linearized equation on ρ, ς takes the following form:

$$\partial_t\rho = \partial_t \begin{bmatrix} \rho_1 \\ \rho_2 \\ \varsigma_1 \\ \varsigma_2 \end{bmatrix} = J\left\{\begin{bmatrix} \rho_2' \\ -\rho_1' \\ \varsigma_2' \\ -\varsigma_1' \end{bmatrix} - \omega\begin{bmatrix} \rho_1 \\ \rho_2 \\ \varsigma_1 \\ \varsigma_2 \end{bmatrix} + g\begin{bmatrix} \rho_1 \\ -\rho_2 \\ \varsigma_1 \\ -\varsigma_2 \end{bmatrix} + 2(v\rho_1 - u\rho_2)g'\begin{bmatrix} v \\ -u \\ 0 \\ 0 \end{bmatrix}\right\} \tag{5.2}$$

where J corresponds to $1/i$:

$$J = \begin{bmatrix} 0 & \boldsymbol{I}_2 \\ -\boldsymbol{I}_2 & 0 \end{bmatrix}.$$

We can write (5.2) as follows:

$$\dot{\rho} = J\mathscr{H}_\omega\rho = \begin{bmatrix} 0 & \boldsymbol{I}_2 \\ -\boldsymbol{I}_2 & 0 \end{bmatrix}\left\{\begin{bmatrix} g-\omega & \partial_x & & \\ -\partial_x & -g-\omega & & \\ & & g-\omega & \partial_x \\ & & -\partial_x & -g-\omega \end{bmatrix} + 2g'\begin{bmatrix} v^2 & -uv & & \\ -uv & u^2 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}\right\}\rho, \tag{5.3}$$

where $\boldsymbol{I}$ is the 2×2 unit matrix.

We define $\mathscr{H}_\pm$ as follows:

$$\mathscr{H}_- = \begin{bmatrix} g-\omega & \partial_x \\ -\partial_x & -g-\omega \end{bmatrix}, \qquad \mathscr{H}_+ = \begin{bmatrix} 2g'v^2 + g - \omega & \partial_x - 2g'vu \\ -\partial_x - 2g'vu & 2g'u^2 - g - \omega \end{bmatrix}.$$

Then the linearization at the solitary wave takes the form

$$\partial_t\begin{bmatrix} \rho \\ \varsigma \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{I}_2 \\ -\boldsymbol{I}_2 & 0 \end{bmatrix}\begin{bmatrix} \mathscr{H}_+ & 0 \\ 0 & \mathscr{H}_- \end{bmatrix}\begin{bmatrix} \rho \\ \varsigma \end{bmatrix} = \begin{bmatrix} 0 & \mathscr{H}_- \\ -\mathscr{H}_+ & 0 \end{bmatrix}\begin{bmatrix} \rho \\ \varsigma \end{bmatrix}.$$

Remark 5.1. A similar block-decomposition was obtained in [CP06, Chu07].

Remark 5.2. Alternatively, to analyze the stability of solitary waves, we need to consider the operator

$$\mathscr{H} = E''(\phi_\omega) - \omega Q''(\phi_\omega).$$

Let $\Phi = (\Phi_1, \Phi_2)$, where $\Phi_1 = \varphi_1 + i\chi_1$, $\Phi_2 = \varphi_2 + i\chi_2$, where φ_j and χ_j are real-valued. We have:

$$Q(\Phi) = \frac{1}{2}\int_{\mathbb{R}} (\varphi_1^2 + \varphi_2^2 + \chi_1^2 + \chi_2^2)\, dx, \qquad Q''(\Phi) = \boldsymbol{I}_4.$$

The energy functional is given by

$$E(\Phi) = \int_{\mathbb{R}} \left(-i\Phi^* \boldsymbol{\alpha} \Phi' + G(\bar{\Phi}\Phi) \right) dx = \int_{\mathbb{R}} \left(\Phi^* \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Phi' + G(|\Phi_1|^2 - |\Phi_2|^2) \right) dx$$

$$= \int_{\mathbb{R}} \left(((\varphi_1 - i\chi_1)(\varphi_2' + i\chi_2') - (\varphi_2 - i\chi_2)(\varphi_1' + i\chi_1')) + G(\varphi_1^2 + \chi_1^2 - \varphi_2^2 - \chi_2^2) \right) dx.$$

For its second derivative, we have:

$$E''(\Phi) = \begin{bmatrix} 0 & \partial_x & 0 & 0 \\ -\partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_x \\ 0 & 0 & -\partial_x & 0 \end{bmatrix} + 2G'' \begin{bmatrix} \varphi_1^2 & -\varphi_1\varphi_2 & \varphi_1\chi_1 & -\varphi_1\chi_2 \\ -\varphi_1\varphi_2 & \varphi_2^2 & -\varphi_2\chi_1 & \varphi_2\chi_2 \\ \varphi_1\chi_1 & -\varphi_2\chi_1 & \chi_1^2 & -\chi_1\chi_2 \\ -\varphi_1\chi_2 & \varphi_2\chi_2 & -\chi_1\chi_2 & \chi_2^2 \end{bmatrix} + G' \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Evaluating this at the solitary wave, $\Phi = \phi_\omega = (v, u)$, with both v, u real, we have:

$$E''(\phi) = \begin{bmatrix} 0 & \partial_x & 0 & 0 \\ -\partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_x \\ 0 & 0 & -\partial_x & 0 \end{bmatrix} + 2G'' \begin{bmatrix} v^2 & -vu & 0 & 0 \\ -vu & u^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + G' \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then

$$\mathscr{H} = E''(\phi) - \omega Q''(\phi) = E''(\phi) - \omega I_4 = \begin{bmatrix} \mathscr{H}_+ & 0 \\ 0 & \mathscr{H}_- \end{bmatrix}.$$

5.1 Spectra of $\mathscr{H}_\pm$

While we are ultimately interested in the spectrum of the operator L_ω, we start by analyzing the specta of $\mathscr{H}_\pm$ which are easier to compute and which will shed some light on the behaviour of the full operator L_ω.

Define

$$m_\pm = 1 \pm \omega.$$

Lemma 5.3. $\sigma_{cont}(\mathscr{H}_\pm) = \mathbb{R} \backslash (-m_+, m_-)$.

Proof. The symbol of $\mathscr{H}_\pm$ at $x \to \pm\infty$ and $\xi = 0$ is given by $\lim_{|x|\to\infty} \sigma(\mathscr{H}_\pm)(x, \xi)|_{\xi=0} = \begin{bmatrix} m_- & 0 \\ 0 & -m_+ \end{bmatrix}$. Its eigenvalues correspond to the edges of the continuous spectrum. □

Lemma 5.4. $\sigma_d(\mathscr{H}_-) \supset \{-2\omega, 0\}$.

Proof. By (3.6), $\mathscr{H}_- \begin{bmatrix} v \\ u \end{bmatrix} = 0$, hence $\phi = \begin{bmatrix} v \\ u \end{bmatrix} \in \ker \mathscr{H}_-$. Moreover, there is a relation

$$(\mathscr{H}_- + 2\omega) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} g + \omega & \partial_x \\ -\partial_x & -g + \omega \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0, \tag{5.4}$$

where again we used (3.6). □

The numerical computations show that there are no other eigenvalues in $\mathscr{H}_-$; this agrees with [Chu07].

Lemma 5.5. $\sigma_d(\mathscr{H}_+) \supset \{-2\omega, 0\}$.

Proof. One can immediately check that

$$\mathscr{H}_+ \begin{bmatrix} v' \\ u' \end{bmatrix} = \begin{bmatrix} 2g'v^2 + g - \omega & \partial_x - 2g'vu \\ -\partial_x - 2g'vu & 2g'u^2 - g - \omega \end{bmatrix} \begin{bmatrix} v' \\ u' \end{bmatrix} = 0.$$

Moreover,

$$(\mathscr{H}_+ + 2\omega) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2g'v^2 + g + \omega & \partial_x - 2g'vu \\ -\partial_x - 2g'vu & 2g'u^2 - g + \omega \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} g + \omega & \partial_x \\ -\partial_x & -g + \omega \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$
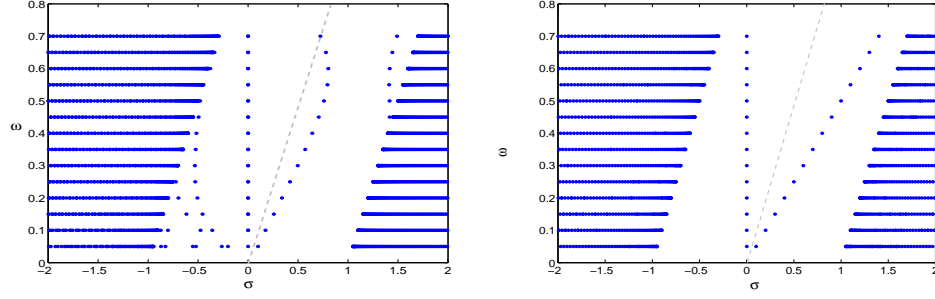
The last equality is due to (3.6). □

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Figure 3: Spectra of H_+ (left) and of H_- (right) versus ω, computed in [Chu07] via Scalapack library on a parallel computer cluster. These plots are courtesy of [Chu07].
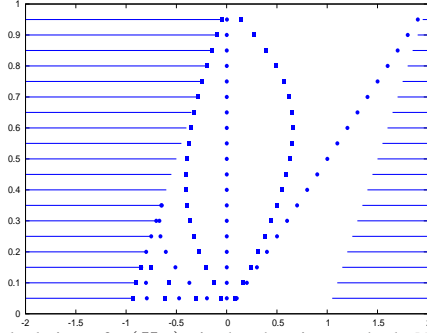

Figure 4: Our recalculation of $\sigma(H_+)$ via the shooting method. Note the presence of the eigenvalue $\lambda = 2\omega$ of $\sigma(H_+)$; the eigenfunction corresponding to $\lambda = 2\omega$ has been found in the explicit form, which seems to confirm our plot. Bifurcation at $\omega \approx 0.5$ is absent, which could be due to a better resolution of the shooting method. We do not know the reason for the discrepancy in computations of $\sigma(H_+)$. We would also like to point out that our recalculation of $\sigma(H_-)$ is in perfect agreement with the right plot on Figure 3 obtained by [Chu07].

The rest of the spectrum of $\mathscr{H}_+$ is computed numerically; see the spectrum of the operator $H_+ := -\mathscr{H}_+$ on Figure 4. (We consider $-\mathscr{H}_+$ for easier comparison with the spectra of H_+ and L_ω from [Chu07] represented on Figure 3.)

5.2 Spectrum of L_ω

Lemma 5.6. *1.*
$$\sigma_{cont}(L_\omega) = (-\infty, -1 + \omega] \cap [1 - \omega, +\infty).$$

2.
$$\ker L_\omega = \mathrm{Span}\left\langle \begin{bmatrix} \partial_x \phi \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \phi \end{bmatrix} \right\rangle.$$

3. $\sigma_d(L) \ni \{\pm 2i\omega\}$.

Proof. Let $\varphi = \begin{bmatrix} u \\ v \end{bmatrix}$. Due to the above results,

$$L \begin{bmatrix} \varphi \\ \pm i\varphi \end{bmatrix} = \begin{bmatrix} & \mathscr{H}_- \\ -\mathscr{H}_+ & \end{bmatrix} \begin{bmatrix} \varphi \\ \pm i\varphi \end{bmatrix} = -2\omega \begin{bmatrix} \pm i\varphi \\ -\varphi \end{bmatrix} = \mp 2\omega i \begin{bmatrix} \varphi \\ \pm i\varphi \end{bmatrix}.$$

To find the continuous spectrum of L_ω, we consider the limit of L_ω as $x \to \pm\infty$, substituting v, u by zeros, g by $g(0) = m = 1$, and ∂_x by $i\xi$. We would also like to find the asymptotics of functions which solve $(L - \lambda)\Psi = 0$ for a particular $\lambda \in \mathbb{C}$. We assume that such a function has the form $\Psi = \begin{bmatrix} r \\ s \end{bmatrix} e^{i\xi x}$. At $x \to \pm\infty$, we have:

$$(L - \lambda) \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -\lambda & H \\ -H & -\lambda \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = 0,$$

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with $H = \begin{bmatrix} m_- & i\xi \\ -i\xi & -m_+ \end{bmatrix}$ coinciding with both $\mathcal{H}_\pm$ at $|x| \to \infty$. Hence $Hs = \lambda r$, $Hr = -\lambda s$, $H^2 r = -\lambda^2 r$,

where $H^2 = \begin{bmatrix} m_-^2 + \xi^2 & -2i\omega\xi \\ 2i\omega\xi & m_+^2 + \xi^2 \end{bmatrix}$. Therefore, λ satisfies

$$0 = \det(H^2 + \lambda^2) = (m_-^2 + \xi^2 + \lambda^2)(m_+^2 + \xi^2 + \lambda^2) - 4\omega^2\xi^2,$$

$$(1 + \omega^2 - 2\omega + \xi^2 + \lambda^2)(1 + \omega^2 + 2\omega + \xi^2 + \lambda^2) - 4\omega^2\xi^2 = 0$$

$$(1 + \omega^2 + \xi^2 + \lambda^2)^2 - 4\omega^2 - 4\omega^2\xi^2 = 0, \qquad 1 + \xi^2 + \omega^2 + \lambda^2 = \pm 2\omega\sqrt{1 + \xi^2},$$

$$1 + \xi^2 + \omega^2 \pm 2\omega\sqrt{1 + \xi^2} = -\lambda^2, \qquad \omega \pm \sqrt{1 + \xi^2} = \pm i\lambda$$

Therefore,

$$\lambda = \pm i(\omega \pm \sqrt{1 + \xi^2}), \qquad \pm i\lambda = \omega \pm \sqrt{1 + \xi^2}, \qquad \sqrt{1 + \xi^2} = \pm\omega \pm i\lambda, \qquad \xi_{\pm\pm} = \pm\sqrt{(\omega \pm i\lambda)^2 - 1}.$$

The continuous spectrum of L_ω (values of λ which correspond to $\xi \in \mathbb{R}$) is

$$\sigma_{cont}(L_\omega) = (-i\infty, -im_-] \cap [im_-, +i\infty).$$

$\square$

Possible resonance points ($\xi = 0$):

$$\lambda^4 + 2\lambda^2(1 + \omega^2) + (\omega^2 - 1)^2 = 0,$$

$$\lambda^2 = -(1 + \omega^2) \pm \sqrt{(1 + \omega^2) - (1 - \omega^2)^2} = -1 - \omega^2 \pm 2\omega,$$

$$\lambda = \pm i(1 \pm \omega).$$

For the eigenfunctions of L_ω corresponding to a particular $\lambda \in \mathbb{C}$ one can take

$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \lambda \begin{bmatrix} 2i\omega\xi \\ m_-^2 + \xi^2 + \lambda^2 \end{bmatrix}, \tag{5.5}$$

which is an eigenvector of H^2 corresponding to $-\lambda^2$.

Remark 5.7. This definition of r gives an eigenvector of H^2 corresponding to $-\lambda^2$ everywhere except at $\lambda = \pm im_-$, where (5.5) vanishes. Then one could take

$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \lambda \begin{bmatrix} m_+^2 + \xi^2 + \lambda^2 \\ -2i\omega\xi \end{bmatrix}, \tag{5.6}$$

which represents an eigenvector of H^2 corresponding to the eigenvalue $-\lambda^2$ everywhere but at $\lambda = \pm im_+$ (where (5.6) vanishes).

Then we use the relation $Hr = -\lambda s$ to find

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = -\begin{bmatrix} m_- & i\xi \\ -i\xi & -m_+ \end{bmatrix}\begin{bmatrix} 2i\omega\xi \\ m_-^2 + \xi^2 + \lambda^2 \end{bmatrix} = -\begin{bmatrix} i\xi(2m_-\omega + m_-^2 + \xi^2 + \lambda^2) \\ 2\omega\xi^2 - m_+(m_-^2 + \xi^2 + \lambda^2) \end{bmatrix}.$$

Denote

$$\xi_{\pm\pm} = \pm\xi_\pm = \pm\sqrt{(\omega \pm i\lambda)^2 - 1} = \pm\sqrt{(-i\lambda \mp \omega)^2 - 1}.$$

Let $\lambda = a + ib$. We assume that both a and b are non-negative. Then

$$\xi_{\pm\pm} = \pm\sqrt{((b - ia) \mp \omega)^2 - 1} = \pm\sqrt{(b \mp \omega)^2 - 2ai(b \mp \omega) - a^2 - 1}.$$

For $\sqrt{z}$, we choose the branch of the square root such that $\mathrm{Re}\,\sqrt{z} \geq 0$. Define

$$\mathbb{C}_{\pm\pm} = \{z \in \mathbb{C}; \ \pm\,\mathrm{Re}\,z \geq 0, \ \pm\,\mathrm{Im}\,z \geq 0\}.$$

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We expect that the eigenvalues emerge from $i(1+\omega)$. Assume that $1 \le b \le 1+\omega$, $0 \le a \ll 1$. Then

$$\xi_{+-} = +\sqrt{(b+\omega)^2 - 2ai(b+\omega) - a^2 - 1} \approx +\sqrt{(1+2\omega)^2 - 2ai(1+2\omega) - 1} \in \mathbb{C}_{+-},$$

$$\xi_{--} = -\xi_{+-} \in \mathbb{C}_{-+}, \qquad \xi_{+-}^2 = \xi_{--}^2 \in \mathbb{C}_{+-},$$

$$\xi_{++} = +\sqrt{(b-\omega)^2 - 2ai(b-\omega) - a^2 - 1} \sim +\sqrt{-a^2 - 2ai} = -i\sqrt{a^2 + 2ai} \in \mathbb{C}_{+-},$$

$$\xi_{-+} = -\xi_{++} \in \mathbb{C}_{-+}, \qquad \xi_{-+}^2 = \xi_{++}^2 \in \mathbb{C}_{--}.$$

The eigenvector corresponding to $\xi = \xi_{\pm\pm}$ is given by

$$\Xi_{\pm\pm} = \begin{bmatrix} r_1 \\ r_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \lambda 2i\omega\xi \\ \lambda(m_-^2 + \xi^2 + \lambda^2) \\ -2m_- i\omega\xi - i\xi(m_-^2 + \xi^2 + \lambda^2) \\ -2\omega\xi^2 + m_+(m_-^2 + \xi^2 + \lambda^2) \end{bmatrix}.$$
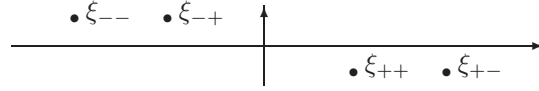
Noting that

$$m_-^2 + \xi_{\pm\pm}^2 + \lambda^2 = (1-\omega)^2 + (\omega \pm i\lambda)^2 - 1 + \lambda^2 = 2\omega^2 - 2\omega \pm 2i\lambda\omega = 2\omega(-m_- \pm 2i\lambda),$$

we have:

$$\Xi_{++} = \begin{bmatrix} i\xi \\ i\lambda - m_- \\ \xi \\ \lambda + im_- \end{bmatrix}, \qquad \Xi_{+-} = \begin{bmatrix} i\xi \\ -i\lambda - m_- \\ -\xi \\ \lambda - im_- \end{bmatrix}, \qquad \Xi_{-+} = \begin{bmatrix} -i\xi \\ i\lambda - m_- \\ -\xi \\ \lambda + im_- \end{bmatrix}, \qquad \Xi_{--} = \begin{bmatrix} -i\xi \\ -i\lambda - m_- \\ \xi \\ \lambda - im_- \end{bmatrix}.$$

ξs in the upper half-plane correspond to waves decaying as $x \to +\infty$

ξs in the lower half-plane correspond to waves decaying as $x \to -\infty$

Figure 5: $\operatorname{Im}\xi_{+\pm} < 0$, hence $e^{i\xi_{+\pm}x}$ decays for $x \to -\infty$. On the other hand $\operatorname{Im}\xi_{-\pm} > 0$, hence $e^{i\xi_{-\pm}x}$ decays for $x \to +\infty$.

Denote by X^- the subspace in $C(\mathbb{R}, \mathbb{C}^4)$ with even first and third components and with odd second and fourth components. Similarly, denote by X^+ the subspace in $C(\mathbb{R}, \mathbb{C}^4)$ with odd first and third components and with even second and fourth components. Then $C(\mathbb{R}, \mathbb{C}^4) = X^+ \oplus X^-$. Noticing that L_ω is invariant in X^+ and X^-, we conclude that all eigenvalues of L_ω always have a corresponding eigenfunction either in X^+ or in X^- (or both). To find eigenvalues of L_ω corresponding to functions from X^+, we proceed as follows:

- For $\lambda \in \mathbb{C}$, construct solutions Ψ_j, $1 \le j \le 4$, to the equation $L\Psi = \lambda\Psi$ with the following initial data at $x = 0$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

  Then $\Psi_1, \Psi_3 \in X^-$, while $\Psi_2, \Psi_4 \in X^+$.

- Pick vectors Ξ_{-+} and Ξ_{--} such that $(L_\omega|_{x=+\infty} - \lambda)\Xi_{-\pm}e^{i\xi_{-\pm}x} = 0$, where $\operatorname{Im}\xi_{-\pm} > 0$.

- Define the analog of the Evans function $E^-(\lambda, R) = \det\left[\Psi_1|_{x=R}, \ \Psi_3|_{x=R}, \ \Xi_{-+}, \ \Xi_{--}\right]$.

- Similarly, define $E^+(\lambda, R) = \det\left[\Psi_2|_{x=R}, \ \Psi_4|_{x=R}, \ \Xi_{-+}, \ \Xi_{--}\right]$.

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- The condition $E^{\pm}(\lambda, R) = 0$ means that a certain linear combination of Ψ_1 and $\Psi_3 \in X^-$ (or, respectively, Ψ_2 and $\Psi_4 \in X^+$) coincides at $x = R$ with a decaying solution of $(L_{x=+\infty} - \lambda)\Psi = 0$. Therefore, if $\lambda \in \sigma_d(L)$, then either $\lim_{R \to +\infty} E^-(\lambda, R) = 0$ or $\lim_{R \to +\infty} E^+(\lambda, R) = 0$ (or both).

Numeric computations do not show any unstable eigenvalues between $\omega = 0.2$ and $\omega = 0.9$ (we do see exact eigenvalues at $\lambda = 0$ and $\lambda = 2\omega i$). On Figure 7, we plot the results of our computations for $\omega = 0.2$, in the vicinity of the point where – presumably – an unstable eigenvalue from [Chu07] (see Figure 6, right) was to be located.

6 Resonances of $\mathscr{H}_+$

We consider the operator

$$\mathscr{H}_+ = \begin{bmatrix} 2g'v^2 + g - \omega & \partial_x - 2g'vu \\ -\partial_x - 2g'vu & 2g'u^2 - g - \omega \end{bmatrix}.$$

6.1 Resonance of $\mathscr{H}_+$ at $\lambda = m_- = m - \omega$

The resonance eigenfunction (M, N) is to satisfy the equation

$$\begin{bmatrix} -N' \\ M' \end{bmatrix} = \begin{bmatrix} -3v^2 + u^2 & 2uv \\ 2uv & -2 + v^2 - 3u^2 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}.$$

We are looking for a solution of the form

$$\lim_{x \to -\infty} \begin{bmatrix} M(x) \\ N(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lim_{x \to +\infty} \begin{bmatrix} M(x) \\ N(x) \end{bmatrix} = \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}.$$

Assuming that $\omega \lesssim 1$ and keeping the principal contribution (see Remark 4.1), we get:

$$-N' = -3v^2 M, \qquad M' = (-2 + v^2)N,$$

$$M'' + (2 - v^2)(3v^2)M = 0.$$

The WKB approximation for a solution is:

$$M(x) \approx \cos\left(\int_{-\infty}^x \sqrt{(2 - v^2(y))3v^2(y)}\, dy\right).$$

The resonance condition takes the form

$$\int_{\mathbb{R}} \sqrt{6v^2}\, dx \approx \sqrt{6} \int_{\mathbb{R}} \sqrt{\mathscr{X}(x)}\, dx = \pi n, \quad n \in \mathbb{Z}. \tag{6.1}$$

Since $\mathscr{X} \approx v^2 \approx 2(m - \omega)e^{-2\kappa|x|}$, where $\kappa = \sqrt{m^2 - \omega^2}$, we have

$$\sqrt{6} \int_{\mathbb{R}} \sqrt{\mathscr{X}}\, dx \approx \sqrt{6} \int_{\mathbb{R}} v(x)\, dx \approx \sqrt{6} \int_{\mathbb{R}} \sqrt{2(m - \omega)}e^{-\kappa|x|}\, dx \approx \sqrt{6} \cdot 2 \cdot \frac{\sqrt{2(m - \omega)}}{\sqrt{m^2 - \omega^2}} = \frac{4\sqrt{3}}{\sqrt{1 + \omega}}.$$

Remark 6.1. Let us check the applicability of the WKB method. As a matter of fact, the applicability condition $\frac{d}{dx}\sqrt{\mathscr{X}(x)} << \mathscr{X}(x)$ does not hold. Let us check the magnitude of the error. We consider the equation

$$-Z'' + \left(-\frac{2}{\cosh^2 x}\right)Z = 0, \qquad Z|_{x=-\infty} = 1,$$

which has the exact solution $Z(x) = -\tanh x$. The WKB method gives the approximate solution

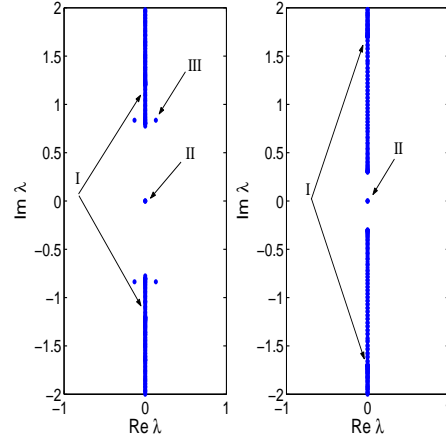$$Z_{WKB}(x) = \cos\left(\int_{-\infty}^x \frac{\sqrt{2}}{\cosh y}\, dy\right).$$

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Figure 6: Spectra of L_ω at $\omega = 0.2$ (left) and $\omega = 0.7$ (right), computed in [Chu07] via Scalapack library on a parallel computer cluster. Note the absence of exact eigenvalues $\lambda = \pm 0.4i$ on the left plot. These plots are courtesy of [Chu07].
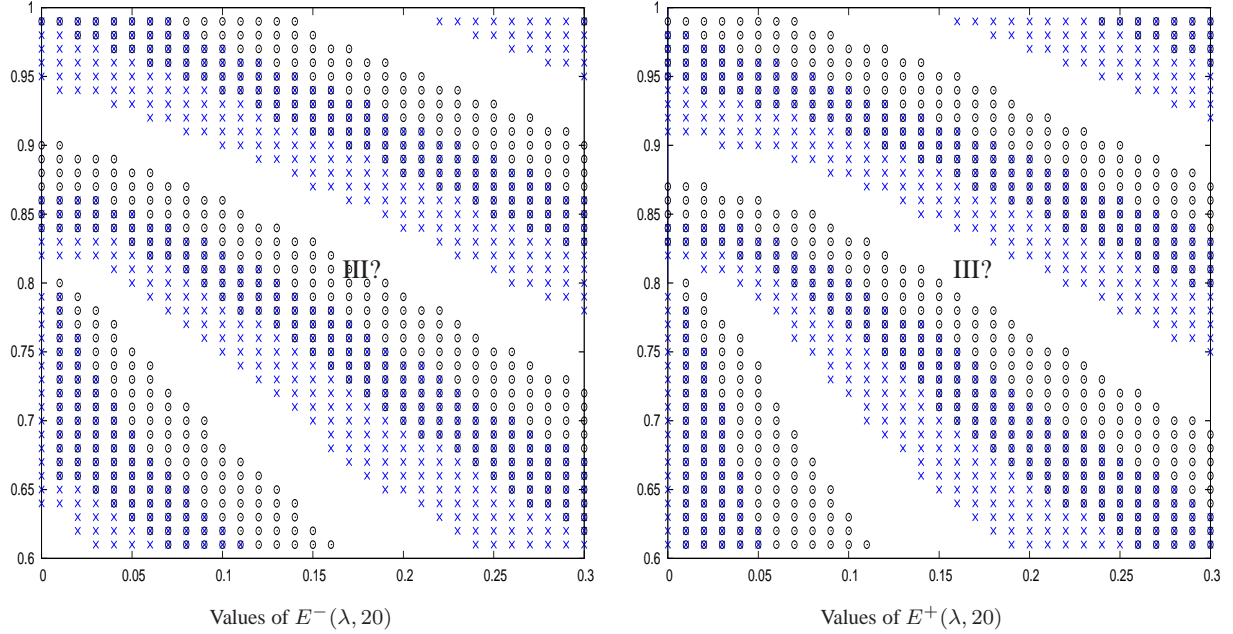

Figure 7: Trying in vain to corroborate an unstable eigenvalue of L_ω at $\omega = 0.2$ by computing the Evans functions $E^\pm(\lambda, R)$ (with $R = 20$) for even-odd-even-odd spinors from X^- (left) and odd-even-odd-even spinors from X^+ (right) in the vicinity of $\lambda \approx 0.15 + 0.8i$. The unstable eigenvalue III ($\lambda \approx 0.15 + 0.8i$) from Figure 6 (left) would be somewhere in the middle of the above plots. The regions where the Evans function takes positive real values are marked with "×", while positive imaginary values are marked with "○". An eigenvalue at a zero of the Evans function would be at the intersection of *borders* of the marked regions. None of such intersections takes place on either plot.

1. The total phase change for $\tanh x$ as x changes from $-\infty$ to $+\infty$ is π, while total phase change of Z_{WKB} is $\int_{\mathbb{R}} \frac{\sqrt{2}}{\cosh y} \, dy = \sqrt{2}\pi$.

2. The asymptotics for $x < 0$, $|x| \gg 1$ for the solution $Z(x) = -\tanh x$ and its approximation $Z_{WKB}(x)$ are

$$Z(x) = -\tanh x = -\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{2x}}{1 + e^{2x}} \approx 1 - 2e^{2x},$$

$$Z_{WKB}(x) \approx \cos\left(\sqrt{2} \int_{-\infty}^{x} \frac{e^y}{2} \, dy\right) = \cos\left(\frac{e^x}{\sqrt{2}}\right) \approx 1 - \frac{e^{2x}}{4}.$$

We see that the WKB asymptotics are imprecise.

The numerical simulations show that the "phase shift" by 2π ($n = 2$ in (6.1)) corresponds to the limit $\omega \to 1$. At the same time, the phase shift which (6.1) gives for $\omega = 1$ is approximately $4\sqrt{3/2} \approx 4.9$. Numerical computations show that the phase shift by 3π ($n = 3$) corresponds to $\omega \approx 0.367$, the phase shift by 4π ($n = 4$) corresponds to $\omega \approx 0.205$. For these values of ω, our approximate condition (6.1) becomes very imprecise since it was derived for $\omega \lesssim 1$.

6.2 Resonance of $\mathscr{H}_+$ at $\lambda = -m_+ = -m - \omega$

The resonance eigenfunction (M, N) is to satisfy the equation

$$\begin{bmatrix} -N' \\ M' \end{bmatrix} = \begin{bmatrix} 2 - 3v^2 + u^2 & 2uv \\ 2uv & v^2 - 3u^2 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}.$$

We are looking for a solution of the form

$$\lim_{x \to -\infty} \begin{bmatrix} M(x) \\ N(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \lim_{x \to +\infty} \begin{bmatrix} M(x) \\ N(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix}.$$

Keeping the principal contribution, we get:

$$-N' = (2 - 3v^2)M, \qquad M' = v^2 N,$$

$$N'' + (2 - 3v^2)(v^2)N = 0.$$

The WKB approximation for a solution is:

$$N(x) \approx \cos\left(\int_{-\infty}^{x} \sqrt{(2 - 3v^2(y))v^2(y)} \, dy\right).$$

The resonance condition takes the form

$$\int_{\mathbb{R}} \sqrt{2v^2} \, dx \approx \sqrt{2} \int_{\mathbb{R}} \sqrt{\mathscr{X}(x)} \, dx = \pi n, \quad n \in \mathbb{Z}. \tag{6.2}$$

7 Resonances of L_ω at $\lambda = im_\pm$

Recall that $m_\pm = m \pm \omega$. The eigenfunctions corresponding to $\lambda = i(m \pm \omega) = im_\pm$ satisfy

$$im_\pm \begin{bmatrix} \rho \\ \varsigma \end{bmatrix} = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{H}_+ & \\ & \mathscr{H}_- \end{bmatrix} \begin{bmatrix} \rho \\ \varsigma \end{bmatrix},$$

or

$$im_\pm \rho = \mathscr{H}_- \varsigma, \qquad im_\pm \varsigma = -\mathscr{H}_+ \rho.$$

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Explicitly,

$$im_{\pm}\rho = \mathcal{H}_{-}\varsigma = \begin{bmatrix} g - \omega & \partial_x \\ -\partial_x & -g - \omega \end{bmatrix}\varsigma, \qquad im_{\pm}\varsigma = -\mathcal{H}_{+}\rho = -\begin{bmatrix} 2g'v^2 + g - \omega & \partial_x - 2g'vu \\ -\partial_x - 2g'vu & 2g'u^2 - g - \omega \end{bmatrix}\rho.$$

$$\begin{cases} \rho_1' = -2g'vu\rho_1 + (2g'u^2 - g - \omega)\rho_2 + im_{\pm}\varsigma_2, \\ -\rho_2' = (2g'v^2 + g - \omega)\rho_1 - 2g'vu\rho_2 + im_{\pm}\varsigma_1, \\ \varsigma_1' = (-g - \omega)\varsigma_2 - im_{\pm}\rho_2, \\ -\varsigma_2' = (g - \omega)\varsigma_1 - im_{\pm}\rho_1. \end{cases}$$

Keeping the main contribution, we get:

$$\begin{cases} \rho_1' = (-m_+ + \mathcal{X})\rho_2 + im_{\pm}\varsigma_2, \\ -\rho_2' = (m_- - 3\mathcal{X})\rho_1 + im_{\pm}\varsigma_1, \\ \varsigma_1' = (-m_+ + \mathcal{X})\varsigma_2 - im_{\pm}\rho_2, \\ -\varsigma_2' = (m_- - \mathcal{X})\varsigma_1 - im_{\pm}\rho_1. \end{cases}$$

7.1 Resonance of L_ω at $\lambda = im_+$

$$\begin{cases} \rho_1' = (-m_+ + \mathcal{X})\rho_2 + im_+\varsigma_2, \\ -\rho_2' = (m_- - 3\mathcal{X})\rho_1 + im_+\varsigma_1, \\ \varsigma_1' = (-m_+ + \mathcal{X})\varsigma_2 - im_+\rho_2, \\ -\varsigma_2' = (m_- - \mathcal{X})\varsigma_1 - im_+\rho_1. \end{cases}$$

We take: $(\rho_1, \rho_2, \varsigma_1, \varsigma_2)|_{x \to -\infty} = (0, 1, 0, -i)$. We differentiate the second and fourth equations and disregard $\mathcal{X}'(x)$:

$$\begin{cases} -\rho_2'' = (m_- - 3\mathcal{X})\rho_1' + im_+\varsigma_1' = (m_- - 3\mathcal{X})((-m_+ + \mathcal{X})\rho_2 + im_+\varsigma_2) + im_+((-m_+ + \mathcal{X})\varsigma_2 - im_+\rho_2) \\ -\varsigma_2'' = (m_- - \mathcal{X})\varsigma_1' - im_+\rho_1' = (m_- - \mathcal{X})((-m_+ + \mathcal{X})\varsigma_2 - im_+\rho_2) - im_+((-m_+ + \mathcal{X})\rho_2 + im_+\varsigma_2), \end{cases}$$

which takes the form

$$\begin{cases} \rho_2'' + m_+^2\rho_2 + (m_- - 3\mathcal{X})(-m_+ + \mathcal{X})\rho_2 + (m_- - 3\mathcal{X})im_+\varsigma_2 + im_+(-m_+ + \mathcal{X})\varsigma_2 = 0 \\ \varsigma_2'' - (m_- - \mathcal{X})im_+\rho_2 - im_+(-m_+ + \mathcal{X})\rho_2 + m_+^2\varsigma_2 + (m_- - \mathcal{X})(-m_+ + \mathcal{X})\varsigma_2 = 0 \end{cases}$$

Disregarding terms with $\mathcal{X}^2$, we get:

$$\begin{bmatrix} \rho_2 \\ \varsigma_2 \end{bmatrix}'' + \left\{ (m_+^2 - m_-m_+)\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} + \mathcal{X}\begin{bmatrix} 3m_+ + m_- & -2im_+ \\ 0 & m_+ + m_- \end{bmatrix} \right\}\begin{bmatrix} \rho_2 \\ \varsigma_2 \end{bmatrix} = 0.$$

Denote $\Gamma_2 = \rho_2 + i\varsigma_2$, $\Upsilon_2 = \rho_2 - i\varsigma_2$. Coupling the above system with $(1, i)$, one gets

$$\Gamma_2'' + \mathcal{X}[3m_+ + m_-, \; -im_+ + im_-]\begin{bmatrix} \rho_2 \\ \varsigma_2 \end{bmatrix} = \Gamma_2'' + \mathcal{X}(m_+ + m_-)\Gamma_2 + 2\mathcal{X}m_+\Upsilon_2 = 0. \tag{7.1}$$

Coupling the above system with $(1, -i)$, one gets

$$\Upsilon_2'' + 2m_+(m_+ - m_-)\Upsilon_2 + \mathcal{X}[3m_+ + m_-, \; -3im_+ - im_-]\begin{bmatrix} \rho_2 \\ \varsigma_2 \end{bmatrix} = \Upsilon_2'' + 2m_+(m_+ - m_-)\Upsilon_2 + \mathcal{X}(3m_+ + m_-)\Upsilon_2 = 0.$$

Thus, $\Upsilon_2 = 0$. The WKB approximation of (7.1) yields $\Gamma_2(x) = C\cos(\int_{-\infty}^{x}\sqrt{2m\mathcal{X}(y)}\,dy)$.

Since the solution (ρ, ς) can be broken into odd and even components, the resonance condition takes the form

$$\int_{\mathbb{R}}\sqrt{2m\mathcal{X}(x)}\,dx = \pi n, \qquad n \in \mathbb{Z}. \tag{7.2}$$

Remark 7.1. Let us note that the resonance condition (6.2) for $\mathcal{H}_+$ at $\lambda = -m_+$ coincides with the resonance condition (7.2) for L_ω at $\lambda = \pm im_+$ (both derived under the same approximations), suggesting similar bifurcation pattern for $\mathcal{H}_+$ at $-m_+$ and for L_ω at $\pm im_+$. Although these criteria in present form are very imprecise, the absence of bifurcations in the spectrum of $\mathcal{H}_+$ indeed matches the absence of bifurcations in the spectrum of L_ω.

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